1

Historical introduction:

From classical electromagnetism to gauge theories

1. Brief history of Maxwell's equations

Magnetism v/s Electricity

1269 : Pierre de Maricourt cuts magnets
1600 : William Gilbert De magnete
1660 : Otto von Guericke sulphur Earth
1687 : Newton's theory of gravitation

End of XVIIth : academies founded
Francis Hauksbee : 1703 builds machines producing electricity

1730 : Grey + Bose conducting / insulating materials
1733 : Dufay two kinds of electricity, vitrous and resinous
1750 : Franklin these two kinds of elec. are of the same nature, conservation of charge
1759 Aepinus (condensator) positive and negative electricities, attraction and repulsion

1740 von Kleist electric commotion
Musschenbroek Leyden jar store sparks

1771: Cavendish charge stays on the outer surface of conductors
In a sphere: less than 1/60 of the charge stays inside
Force in $\frac{1}{r^\alpha}$: $\alpha = 2$ within 2%

1780 Coulomb studies torsion of threads
Coulomb's scale

1785 Mémoires sur l'électricité et le magnétisme
3 numerical values : $\frac{1}{r^2}$ law.

1810 Poisson does not try to explain the nature of electricity

$$-\Delta V = \rho = \operatorname{div} \vec{E}$$

Electrodynamics

1790 Galvani's experiments in Bologne with frogs
Sparks produce contraction
two metals also
even one metal, or no metal at all
Controversy with Volta
Galvani : animal electricity
Volta : metallic electricity
Volta studies torpedo fish
1800 Volta's pile

Analogy pile / magnet, electricity / magnetism

Ritter: natural philosophy

1820: Oersted's experiment
helicoidal lines of force
not a Newtonian theory

1821 Arago reproduces the experiment in Geneva in front of Ampère

Biot : force exerted by a magnet on a wire with current
Thinks that a current is a collection of magnets

Ampère thinks that a magnet is a bunch of currents
Idea of interaction of currents

$$\operatorname{curl} \vec{B} = \vec{j}$$

1830 After unsuccessful attempts by Colladon in Geneva, and using electro-magnets built by Henry, Faraday produces current from magnets.

$$\operatorname{curl} \vec{E} - \frac{\partial \vec{B}}{\partial t} = 0$$

The fourth equation:

$$\operatorname{div} \vec{B} = 0$$

unclear origin (Kelvin? Another Thomson? Ampère himself?)

1864 Maxwell - synthetises
- adds the term

$$\operatorname{curl} \vec{B} + \frac{\partial \vec{E}}{\partial t} = \vec{j}$$

for the sake of conservation of charge
- proposes that light is an electromagnetic phenomenon.

1700 → 1900 unification of
electricity
magnetism
light
heat

$$\begin{cases} \operatorname{div} \vec{E} = \rho \\ \operatorname{curl} \vec{E} - \frac{\partial \vec{B}}{\partial t} = 0 \\ \operatorname{div} \vec{B} = 0 \\ \operatorname{curl} \vec{B} + \frac{\partial \vec{E}}{\partial t} = \vec{j} \end{cases}$$

Lorentz force $q\vec{E} + q\vec{v} \wedge \vec{B}$

2. The geometry of Maxwell's equations

Minkowski introduces

$$E = E_x\, dx + E_y\, dy + E_z\, dz$$

$$B = B_x\, dy \wedge dz + B_y\, dz \wedge dx + B_z\, dx \wedge dy$$

$$F = B - dt \wedge E$$

differential forms on space-time $\mathbb{R}^4$

$dF = 0$ homogeneous ME

Metric $dt^2 - dx^2 - dy^2 - dz^2$,

orientation $dt \wedge dx \wedge dy \wedge dz$

$*$ Hodge operator $*F(\vec{E}, \vec{B}) = F(-\vec{B}, \vec{E})$

$$J = \rho\, dx \wedge dy \wedge dz + j_x\, dt \wedge dy \wedge dz + j_y\, dt \wedge dz \wedge dx + j_z\, dt \wedge dx \wedge dy$$

$d{*F} = J$ inhomogeneous ME

$$\begin{cases} dF = 0 \\ d{*F} = J \end{cases}$$

Lagrangian form of the equations:

$F = dA$
- locally
- $A \rightsquigarrow A + d\varphi$

$$\mathcal{L}(A) = \frac{1}{2} F \wedge *F + A \wedge J$$

Why is the Lagrangian a function of A and not F?

- A is observable, up to gauge-transformation (Bohm-Aharonov)
- F does not always determine A up to gauge transformations (topology; Gribov ambiguity)

Quantum mechanics of a non-relativistic charged particle

mass m, charge q

$$\mathcal{L}(\vec{r}, \vec{v}) = \frac{1}{2} m \|\vec{v}\|^2 - qV + q\vec{v} \cdot \vec{A}$$

$$A = -V\, dt + A_x\, dx + A_y\, dy + A_z\, dz$$

$$\mathcal{H}(\vec{r},\vec{p}) = \frac{1}{2m}(\vec{p}-q\vec{A})^2 + qV$$

$$\vec{p} = m\vec{v} + q\vec{A}$$

Schrödinger equation:

$$(\partial_t + iH)\psi = 0$$

with $H = -\frac{1}{2m}(\vec{\nabla} - iq\vec{A})^2 + qV$

Replace A by $A + d\varphi$

ie $\left| \begin{array}{l} V \text{ by } V - \partial_t\varphi \\ \vec{A} \text{ by } \vec{A} + \vec{\nabla}\varphi \end{array} \right.$

H becomes $H_\varphi = -\frac{1}{2m}(\vec{\nabla} - iq\vec{A} - iq\vec{\nabla}\varphi) + q(V - \partial_t\varphi)$

<u>Fact</u>: $\partial_t + iH_\varphi = e^{iq\varphi}(\partial_t + iH)e^{-iq\varphi}$

A change of gauge modifies the Hamiltonian and must be accompanied by a change of the wave functions.

Principal $U(1)$-bundle over space-time (the set of values of norm 1 of the wave functions of a particle of charge 1)

A: connection
F: curvature of A

Non-abelian gauge theories

$$\begin{array}{c} P \circlearrowleft G \\ \downarrow \\ M \end{array} \qquad A,\ F = dA + [A,A]$$

$$\mathcal{L}(A) = \frac{1}{2}\langle F \wedge *F\rangle + \langle A \wedge J\rangle$$

$\leftarrow$ $\mathfrak{g}$-valued current 3-form ?

Chromohydrodynamics

3. <u>Feynman's path integrals</u>

Particle in a potential

$$H = -\frac{\hbar^2}{2m}\Delta + V$$

Want to solve $i\hbar\partial_t\psi = H\psi$, ie

$\partial_t\psi = (\alpha\Delta + \beta V)\psi$ with $\alpha = \frac{i\hbar}{2m}$, $\beta = \frac{1}{i\hbar}$

$$\psi(t,\vec{x}) = \left(e^{t\alpha\Delta + t\beta V}\psi(0,\cdot)\right)(\vec{x})$$

Formally,

$$(e^{s\Delta}f)(x) = \int f(x+y)\, e^{-\frac{y^2}{4s}} \frac{dy}{\sqrt{4\pi s}}$$

Also, $e^{A+B} = \lim_{n\to\infty}\left(e^{\frac{A}{n}}e^{\frac{B}{n}}\right)^n$.

One finds, formally,

$$\psi(1,\vec{x}) = \lim_{n\to\infty}\left(\left(e^{\frac{\alpha}{n}\Delta}\, e^{\frac{\beta}{n}V}\right)^n \psi(0,\cdot)\right)(\vec{x})$$

$$= \lim_{n\to\infty}\int e^{\frac{\beta}{n}V(x+y)}\, e^{-\frac{n}{4\alpha}y^2}\left(\left(e^{\frac{\alpha}{n}\Delta}\, e^{\frac{\beta}{n}V}\right)^{n-1}\psi(0,\cdot)\right)(x+y)\,\frac{dy}{\sqrt{4\pi\frac{\alpha}{n}}}$$

$$= \lim_{n\to\infty}\int e^{\frac{\beta}{n}V(x+y_1)+\frac{\beta}{n}V(x+y_1+y_2)+\cdots+\frac{\beta}{n}V(x+y_1+\cdots+y_n)}\, e^{-\frac{n}{4\alpha}(y_1^2+\cdots+y_n^2)}\,\psi(0,x+y_1+\cdots+y_n)\,\frac{dy_1\cdots dy_n}{\left(4\pi\frac{\alpha}{n}\right)^{\frac{n}{2}}}$$

$$\propto \int_{\{q:[0,1]\to\mathbb{R},\ q(0)=x\}} e^{\int_0^1 -\frac{1}{4\alpha}\dot{q}(t)^2+\beta V(q(t))\,dt}\,\psi(0,q(1))\,Dq$$

$$= \int e^{\frac{i}{\hbar}\int_0^1 \frac{m\dot{q}(t)^2}{2}-V(q(t))\,dt}\,\psi(0,q(1))\,Dq$$

The amplitude of presence at point z is

$$\int_{\{q:[0,1]\to\mathbb{R},\ q(0)=x,\ q(1)=z\}} e^{\frac{i}{\hbar}S(q)}\,Dq\,.$$

In the context of gauge theories, these integrals become

$$\int_{\substack{\text{some}\\ \text{subset of } \mathcal{A}}} e^{\frac{i}{\hbar}S(A)}\,DA\,.$$

Stochastic quantisation: make sense of

$$d\mu(A) = \frac{1}{Z}\, e^{-\frac{1}{g}S(A)}\,DA$$

as a probability measure.

Two-dimensional Yang-Mills

Review of EM: Maxwell equations, potentials, force law, action

Gauge fields mediate interaction between fundamental constituents

EM field is described through a potential, a 1-form A on 4-dim. spacetime Σ.

field strength: a 2-form

$$F = dA$$

$$F = B - dt \wedge E$$

$$B = \vec{B} \cdot (dy \wedge dz,\ dz \wedge dx,\ dx \wedge dy)$$

$$E = \vec{E} \cdot (dx, dy, dz)$$

The force exerted by the field on a point charge e moving at velocity v depends linearly on v and is Minkowski-orthogonal to v.

Force: mass $\times$ rate of change of momentum (which is a covector).
Lorentz force law:

$$\text{force} = e\ \iota_v F$$

Maxwell equations

$$\begin{cases} dF = 0 \\ d{*}F = \mu_0 J \end{cases}$$

The electromagnetic field has the ability to do work on charges and currents. It thus has energy.

Maxwell worked out the energy density of the field

$$u = \frac{\varepsilon_0}{2}\left(\|\vec{E}\|^2 + c^2\|\vec{B}\|^2\right)$$

Momentum of the field: Poynting vector

$$\vec{P} = \frac{\mu_0}{2}\vec{E}\times\vec{B}$$

Write $A = -\frac{1}{c}\phi\, dt + \vec{A}\cdot(dx, dy, dz)$

$$L_{EM}\left((\phi,\vec{A}),(\dot{\phi},\dot{\vec{A}})\right) =$$

$$\frac{\varepsilon_0}{2}\int_{\mathbb{R}^3}\left[\left(\|\vec{E}\|^2 - c^2\|\vec{B}\|^2\right) - \left(\rho\phi - \vec{J}\cdot\vec{A}\right)\right] dvol$$

$$\vec{E} = -\nabla\phi - \dot{\vec{A}} \quad , \quad \vec{B} = \nabla\times\vec{A}$$

$$S_{EM} = \int_{\mathbb{R}} L_{EM}\left((\phi,\vec{A}),(\dot{\phi},\dot{\vec{A}})\right) dt$$

This works out as

$$S_{EM}(A) = \int_{\Sigma} \mathcal{L}_{EM}(A)$$

$$\mathcal{L}_{EM}(A) = -\frac{c^2\varepsilon_0}{2} F\wedge{*}F - A\wedge J$$

(The sign matters if one is going to look for the Hamiltonian and understand it as energy)

(There is a formalism for non-orientable spacetimes, using densities)

<u>Quantization for a particle in the EM field and gauge theory.</u>

Quantum mechanics puts a restriction on the behavior of the field strength F

$$\frac{e}{h}\int_{\sigma} F \in \mathbb{Z}$$

for every closed oriented 2-manifold σ sitting in spacetime.
(Dirac monopole charge quantization condition).

This is precisely the condition for

$$i\frac{e}{\hbar}F$$

to be the curvature for a connection

$i\frac{e}{\hbar}A$ on a $U(1)$-bundle over Σ.

This geometric feature of the electromagnetic field makes it a gauge theory ie a theory of connections.

Gauge invariance.

ψ Schrödinger wave function for a particle charge e in potential A

Compute with

$$\psi \quad \text{and} \quad A$$

or

$$\psi_\theta = e^{\frac{ie}{\hbar}\theta}\psi \quad \text{and} \quad A^\theta = A + d\theta$$

θ: 'angle of rotation' in 'charge space'.

Then $\left(\partial_j - i\frac{e}{\hbar}A_j\right)\psi \mapsto \left(\partial_j - i\frac{e}{\hbar}A_j^\theta\right)\psi_\theta$

$$F = dA = dA^\theta$$

Since it is $i\frac{e}{\hbar}A$ that is actually a connection form (has dimension $[L^{-1}]$), let us rewrite this as A and keep $F = dA$.

$$S_{rm\,EM}(A) = -k\int_\Sigma F^A \wedge *F^A$$

Nucleons

1932: Heisenberg suggested that the proton and the neutron could be viewed as different states of the same particle.

"isospin": they 'are' different states of isospin of the same entity

As far as isospin is concerned, the nucleon wave function ψ is something on which $SU(2)$ matrices should be able to act. (Just as $U(1)$ acts on the traditional wave function for an electrically charged particle).

Yang-Mills: extending to isotopic gauge invariance

ψ the Schrödinger wave function for a particle with isotopic spin moving in the field that propagates the interaction between particles with isotopic spin.

Yang and Mills (1954) suggested a local gauge invariance principle

$$\psi \quad \text{and} \quad A$$

or

$$\psi_U = U\psi \quad \text{and} \quad A^U = UAU^{-1} - (dU)U^{-1}$$

U: $SU(2)$-valued function on spacetime.

A: 1-form with values in 2x2 skew-Hermitian matrices.

Field strength

$$F^A = dA + A \wedge A$$

Classical field configurations are extrema of the Yang-Mills action

$$S_{YM}(A) = -\frac{1}{2g^2} \int_\Sigma \mathrm{Tr}(F^A \wedge * F^A)$$

Euler-Lagrange equations: YM equations.

Gauge fields as connection forms

1-form A on Σ with values in the Lie algebra $L(G)$ of a compact Lie group $G \subset U(N)$.

$v \in T_m\Sigma \qquad A(v) \in L(G)$

Connection, or a gauge field

Set of all connections is an infinite dim. vector space

$$\mathcal{A}$$

Metric structure

Ad-inv.

$$\langle A, B \rangle = \int_\Sigma \langle A, B \rangle_{LG} \, d\mathrm{vol}$$

Riemannian

A gauge transformation $\phi : \Sigma \to G$

$\mathcal{G}$ group under pointwise mult.

$$A^\phi = \phi^{-1} A \phi + \phi^{-1} d\phi$$

$\mathcal{A}/\mathcal{G}$ quotient space

$\mathcal{G}_0$: those $\phi \in \mathcal{G}$ s.t. $\phi(o) = I$ for some basepoint o.

Parallel transport

Connection A, smooth path $c : [0,1] \to \Sigma$

$$[0,1] \longrightarrow G$$
$$t \longmapsto g_t$$

$$\frac{dg_t}{dt} = -A(c'(t))\, g_t$$
$$g_0 = I$$

$g_1 \in G$ is called the holonomy of A around c

$$h_c(A) = h(c; A) = g_1$$

$\mathrm{Tr}(h_c(A))$ Wilson loop variable

Non-abelian gauge theory: quantum functional integral

Quantizing the gauge field itself requires (in one approach) using a functional integral measure

$$\frac{1}{Z_g} e^{-S_{YM}(A)} \, DA$$

$$\frac{1}{Z_g} \int_{\mathcal{A}} f(A) \, e^{-S_{YM}(A)} \, DA$$

Simplifying the quartic

$$S_{YM}(A) = \| dA + A \wedge A \|^2$$

which is quartic in A and the problem is difficult.

Work with $\mathcal{A}/\mathcal{G}$: leads to a simplification.

YM on $\mathbb{R}^2$ is Gaussian

$$\mathcal{A}/\mathcal{G}_0 \simeq \quad A = A_x \, dx + A_y \, dy$$

with $A_y = 0$.

$$F^A = dA + 0$$

This makes our functional integral measure have a very convenient appearance:

$$\frac{1}{Z_g} e^{-\frac{1}{2g^2}\|f\|^2} \, Df$$

No useful form of Lebesgue in infinite dimensions.
But useful, very useful Gaussian.

The Yang-Mills measure for gauge theory on $\mathbb{R}^2$ is Gaussian measure on functions

$$f : \mathbb{R}^2 \to L(G)$$

Stochastic geometry.

Consider a path

$$c : [0,1] \to \mathbb{R}^2$$
$$t \mapsto (t, y(t))$$

$$dg_t = -A(c'(t)) \, g_t \, dt$$

Now that A is stochastic, this can be reinterpreted as a Stratonovich SDE.

If c is a nice loop in $\mathbb{R}^2$, the holonomy

$$h_c(A)$$

as a function of the stochastic A, is a random variable with values in G.

Its distribution has density

$$Q_{g^2|S|}(x)$$

where $|S|$ is the area enclosed by the loop c.

$$\partial_t Q_t(x) = \frac{1}{2} \Delta Q_t(x).$$

Loop expectation values.

Theorem • c simple closed loop in $\mathbb{R}^2$ enclosing area S : holonomy h_c distributed according to

$$Q_{g^2|S|}(x)\,dx$$

• non-overlapping loops are mutually independent.

Single loop for $U(N)$

$$\mathbb{E}[\mathrm{Tr}_N\, h_c] = e^{-Ng^2\frac{S}{2}}$$

Compact manifolds

2-dim. Riem. mfd Σ

Build Σ from a disk by identifications

Conditional probability measure : holonomy along this arc equals holonomy along that arc.

Distribution of the random variables

$$h_{c_1}, \dots, h_{c_n}$$

determine all that is of interest for the YM measure.

Stochastic holonomy fields

Σ 2-dim. Riem. mfd

$L_0(\Sigma)$ set of rectifiable loops

h_c holonomy, with f.d. distr.

Loop expectation value on the sphere

c on the sphere S^2

h_c Brownian bridge

$$\mathbb{E}[f(h_c)] = \frac{1}{Q_{|S^2|}(e)} \int_{G^2} f(x)\, Q_{|S|}(x)\, Q_{|S^c|}(x^{-1})\, d$$

General configuration of loops

Statistical physics flavor.
Graphs on the surface.

Freezing the measure

$$d\mu_g(A) = \frac{1}{Z_g} e^{-\frac{1}{2g^2}\|F^A\|^2}\, DA$$

$g^2 \to 0$ μ_g freezes on the space of flat connections

Large N limit

$U(N)$, $N \to \infty$, holding Ng^2 constant.

3

A connection with random matrix theory.

U(N) Yang-Mills theory on $\mathbb{R}^2$

$\mathbb{R}^2$ U(N) Yang-Mills measure

$$d\mu_{YM}(A) = \frac{1}{Z_g} e^{-\frac{1}{2g^2} S_{YM}(A)} DA$$

ℓ loop $h(\ell, A) \in U(N)$ holonomy along ℓ

A random under μ_{YM}:

$h(\ell, A) = h_\ell$ U(N)-valued r.v.

①  $h_\ell \sim Q_{g^2 s}(U)\, dU$

heat kernel Haar meas.

= dist. of $B_{g^2 s}$ $(B_t)_{t \geq 0}$ BM on U(N)

②  $h_{\ell_1}, h_{\ell_2}, h_{\ell_3}$ indep.

③  $h_{\ell_1 \ell_2} = h_{\ell_1} h_{\ell_2}$

"Brownian motion on U(N) indexed by loops, with area playing the role of time, and with a property of multiplicativity"

The "free group" of loops

loops can be concatenated

notion of cancellation (backtracking)

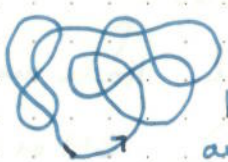
the group of loops "is" free on the set of elementary loops (lassos):

any reasonable loop can be written as a word in non-overlaping lassos

Ex:

h_ℓ has the distribution of $B_s \cdot \tilde{B}_t^2$

index · BMs on $U(N)$



more complicated, but algorithmically and analytically tractable

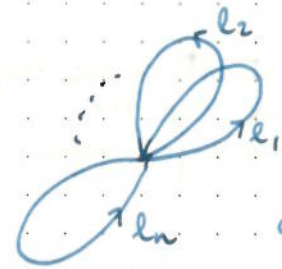
Wilson loop expectations

What are the numerical quantities of interest ?

Gauge-invariant quantities

$$(A, \ell) \longmapsto h(\ell, A)$$

$$(A^U, \ell) \longmapsto h(\ell, A^U) = U(\ell(0))\, h(\ell, A)\, U(\ell(0))^{-1}$$



all the holonomies are conjugated by the same element of $U(N)$, namely the value of U at the common basepoint.

Basic GI quantity: $h_{\ell_1}, \dots, h_{\ell_n}$ up to simultaneous conjugation.

- Unitary matrix U up to conjugation: spectrum of U
- $U_1, \dots, U_n$ up to simultaneous conj.: spectra of $U_1, \dots, U_n$ + relative pos$^{\underline{n}}$ of eigenspaces

<u>Theorem</u> If for every word w in n letters and their inverses one has

$$\mathrm{Tr}\big(w(U_1, \dots, U_n)\big) = \mathrm{Tr}\big(w(U'_1, \dots, U'_n)\big),$$

then there exists $V \in U(N)$ s.t. $\forall i = 1 \dots n$, $U'_i = V U_i V^{-1}$.

The basic numerical quantity is

$$\mathrm{Tr}\big(w(h_{\ell_1}, \dots, h_{\ell_n})\big) = \mathrm{Tr}\big(h_{w(\ell_1, \dots, \ell_n)}\big)$$

Wilson loops expectations:

$$\mathbb{E}\big[\mathrm{Tr}(h_{\ell_1}) \dots \mathrm{Tr}(h_{\ell_n})\big]$$

Using · the "free group" structure of loops
· the description of the YM meas. on $\mathbb{R}^2$ (with a lot of independence),

this becomes

$$\mathbb{E}\left[\mathrm{Tr}\left(w_1(B^{(1)}_{t_1},\dots,B^{(p)}_{t_p})\right)\dots \mathrm{Tr}\left(w_n(B^{(1)}_{t_1},\dots,B^{(p)}_{t_p})\right)\right]$$

indep. BM

words in n letters and their inverses

Brownian motion on $U(N)$

Laplacian on $U(N)$

$\mathfrak{u}(N)$ $N\times N$ skew-Hermitian matrices

$\langle X,Y\rangle = N\,\mathrm{Tr}(X^*Y) = -N\,\mathrm{Tr}(XY)$

$(g^2=\frac{1}{N})$

ONB : $\frac{1}{\sqrt{2N}}\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}, \frac{1}{\sqrt{2N}}\begin{pmatrix} & i \\ i & \end{pmatrix}, \frac{1}{\sqrt{N}}\begin{pmatrix} i & \\ & \end{pmatrix}$

$(X_k)_{k=1..N^2}$

$f: U(N)\longrightarrow \mathbb{R}$ smooth

$$(\Delta f)(U) = \sum_{k=1}^{N^2} \frac{d^2}{dt^2}\Big|_{t=0} f(Ue^{tX_k})$$

Ex $f(U)=\mathrm{Tr}(U)$

$$(\Delta f)(U) = \sum_{k=1}^{N^2}\mathrm{Tr}(UX_k^2) = \mathrm{Tr}\left(U\sum_{k=1}^{N^2}X_k^2\right)$$

(C1) $\displaystyle\sum_{k=1}^{N^2} X_k^2 = -I_N$

$\Delta\,\mathrm{Tr} = -\mathrm{Tr}$

Brownian motion on $U(N)$: Markov process on $U(N)$ with generator $\frac{1}{2}\Delta$, issued from I_N.

$(B_t)_{t\geq 0}$, $B_0 = I_N$.

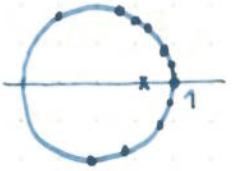
$f: U(N)\longrightarrow\mathbb{R}$

$$\partial_t\,\mathbb{E}\left[f(B_t)\right] = \mathbb{E}\left[\tfrac{1}{2}(\Delta f)(B_t)\right]$$

$$\mathbb{E}\left[f(B_0)\right] = f(I_N)$$

Example $\mathrm{tr} = \frac{1}{N}\mathrm{Tr}$

$$\mathbb{E}\left[\mathrm{tr}(B_t)\right] = e^{-\frac{t}{2}}$$



Brownian motion on $U(N)$ and walk on S_n

$\mathbb{E}\left[\mathrm{tr}(B_t^2)\right] = ?$

$f(U) = \mathrm{tr}(U^2)$

$$(\Delta f)(U) = \sum_{k=1}^{N^2}\frac{d^2}{dt^2}\Big|_{t=0}\mathrm{tr}(Ue^{tX_k}Ue^{tX_k})$$

$$= 2\,\mathrm{tr}\left(U^2\sum_{k=1}^{N^2}X_k^2\right) + 2\sum_{k=1}^{N^2}\mathrm{tr}(UX_kUX_k)$$

(C2) $\sum_{k=1}^{N^2} \mathrm{tr}(A X_k B X_k) = -\mathrm{tr}(A)\,\mathrm{tr}(B)$

$$(\Delta f)(U) = -2\,\mathrm{tr}(U^2) - 2\,\mathrm{tr}(U)^2$$

$$\partial_t \mathbb{E}[\mathrm{tr}(B_t^2)] = -\mathbb{E}[\mathrm{tr}(B_t^2)] - \underbrace{\mathbb{E}[\mathrm{tr}(B_t)^2]}_{=?} \quad (*)$$

$$g(U) = \mathrm{tr}(U)^2$$

$$(\Delta g)(U) = -2\,\mathrm{tr}(U)^2 + 2\sum_{k=1}^{N^2} \mathrm{tr}(U X_k)\,\mathrm{tr}(U X_k)$$

(C3) $\sum_{k=1}^{N^2} \mathrm{tr}(A X_k)\,\mathrm{tr}(B X_k) = -\frac{1}{N^2}\,\mathrm{tr}(AB)$

$$(\Delta g)(U) = -2\,\mathrm{tr}(U)^2 - \frac{2}{N^2}\,\mathrm{tr}(U^2)$$

$$\partial_t \mathbb{E}[\mathrm{tr}(B_t)^2] = -\mathbb{E}[\mathrm{tr}(B_t)^2] - \frac{1}{N^2}\mathbb{E}[\mathrm{tr}(B_t^2)] \quad (**)$$

Solve $(*)$, $(**)$:

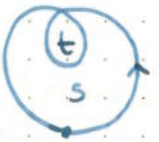
$$\mathbb{E}[\mathrm{tr}(B_t)^2] = e^{-t}\left(\mathrm{ch}\,\frac{t}{N} - \frac{1}{N}\,\mathrm{sh}\,\frac{t}{N}\right)$$

$$\mathbb{E}[\mathrm{tr}(B_t^2)] = e^{-t}\left(\mathrm{ch}\,\frac{t}{N} - N\,\mathrm{sh}\,\frac{t}{N}\right)$$

$\underline{N \to \infty}$: $\mathbb{E}[\mathrm{tr}(B_t)^2] \longrightarrow e^{-t}$

$$= \lim_{N\to\infty} \mathbb{E}[\mathrm{tr}(B_t)]^2$$

$$\mathbb{E}[\mathrm{tr}(B_t^2)] \longrightarrow e^{-t}(1-t).$$



$$\mathbb{E}[\mathrm{tr}(h_\ell)] = \mathbb{E}[\mathrm{tr}(B_s \cdot \tilde{B}_t^2)]$$

As $N \to \infty$, asymptotic freeness (in the sense of free probability) allows one to separate the two BMs:

$$\lim_{N\to\infty} \mathbb{E}[\mathrm{tr}(h_\ell)] = \lim_{N\to\infty} \mathbb{E}[\mathrm{tr}(B_s)] \cdot \lim_{N\to\infty} \mathbb{E}[\mathrm{tr}(\tilde{B}_t^2)]$$

$$= e^{-\frac{s}{2} - t}(1-t).$$

<u>In general</u>:

- (C1), (C2), (C3) can be subsumed in

$$\sum_{k=1}^{N^2} X_k \otimes X_k = -\frac{1}{N} T$$

ie for all bilinear form β on $\mathbb{C}^N$, all $x, y \in \mathbb{C}^N$,

$$\sum_{k=1}^{N^2} \beta(X_k x, X_k y) = -\frac{1}{N}\beta(y, x)$$

Flip = transposition, appearance of permutations.

- σ permutation with cycle lengths $m_1, \dots, m_r$:

$$p_\sigma(U) = \mathrm{tr}(U^{m_1}) \dots \mathrm{tr}(U^{m_r}) \quad \text{(notation)}$$

- σ permutation, $\tau = (i\,j)$ transposition

$\sigma\tau$ has 1 more / less cycle if i,j are in the same / different cycles of σ

$$\frac{1}{2}\Delta p_\sigma(U) = -\frac{n}{2} p_\sigma(U) - \sum_{\substack{(i\,j)\\ \text{same cycle}}} p_{\sigma(i\,j)}(U)$$

$$-\frac{1}{N^2} \sum_{\substack{(i\,j)\\ \text{different cycles}}} p_{\sigma(i\,j)}(U)$$

The $n!$ functions

$$\left(t \mapsto \mathbb{E}\left[p_\sigma(B_t)\right]\right)_{\sigma \in S_n}$$

satisfy a first order linear diff. sys.

This system can be solved for finite N.

As $N \to \infty$, the system becomes simpler.

<u>Theorem</u> · $\mathbb{E}\left[\operatorname{tr}(B_t^{m_1}) \dots \operatorname{tr}(B_t^{m_r})\right] =$

$$\mathbb{E}\left[\operatorname{tr}(B_t^{m_1})\right] \dots \mathbb{E}\left[\operatorname{tr}(B_t^{m_r})\right] + O\left(\frac{1}{N^2}\right)$$

(factorisation property)

· $\lim_{N\to\infty} \mathbb{E}\left[\operatorname{tr}(B_t^n)\right] = e^{-\frac{nt}{2}} \sum_{k=0}^{n-1} \frac{(-t)^k}{k!} \binom{n}{k+1} n^{k-1}$

(moments of the asymp. dist. of eigenvalues)

<u>Wigner's thm</u> $Z_1, \dots, Z_{N^2}$ iid $\mathcal{N}(0,t)$

$$M = \sum_{k=1}^{N^2} Z_k (iX_k)$$ Gaussian Hermitian matrix

The eigenvalues of M are dist. according to

$$d\sigma_t(x) = \frac{1}{2\pi t}\sqrt{4t - x^2}\, \mathbb{1}_{[-2\sqrt{t},\, 2\sqrt{t}]}(x)\, dx$$

<u>Here</u> there is a prob. meas. on $\mathbb{U}$, where $\mathbb{U} = \{z \in \mathbb{C} : |z| = 1\}$, s.t. the eigenvalues of B_t are (almost) distributed according to ω_t.

$$\operatorname{Supp}(\omega_t) = \begin{cases} \{e^{i\theta} : |\theta| \leq \beta(t)\}, & \beta(t) = \frac{1}{2}\int_0^t \sqrt{\frac{4-s}{s}}\, ds \quad \text{if } t < 4 \\ \mathbb{U} & \text{if } t \geq 4 \end{cases}$$

Small t: $\beta(t) \simeq 2\sqrt{t}$.

4

Three-dimensional Chern-Simons theory.

Study of infinite dimensional integrals of the type

$$\int_{\mathcal{A}} f(A)\, e^{-\beta S(A)}\, DA$$

is rich with challenges and questions arising both from physics and mathematic

Cases where this integral represents a quantity of geometric or topological meaning that can be understood in other ways.

Sometimes a limit $\ldots \to 0, \infty$ is involved.

One usually treats the left side formally and extracts insights from this formal integral.

- Perturbative
- Non-perturbative $\quad$ def$^{\underline{no}}$

Yang-Mills

$$\frac{1}{Z_g} \int_{\mathcal{A}} f(A)\, e^{-\frac{1}{2g^2} S_{YM}(A)}\, DA$$

Chern-Simons

$$\int_{\mathcal{A}} f(A)\, e^{ik\, CS(A)}\, DA$$

At a heuristic level, $\mathcal{D}A$ requires a choice of metric (or something similar) on $\mathcal{A}$.

V infinite dim. space

$$\int_V f(x)\, e^{-\beta S(x)}\, dx$$

is a linear functional

$$\Phi : f \mapsto \Phi(f)$$

Formal calculations specify what $\Phi(f)$ should be for some good class of functions.

Φ should have continuity properties

Φ might come from integration w.r.t. a measure, or be a distribution

V infinite dim. real vector space

$$\Phi(e^{i\langle x,\cdot\rangle}) = e^{-\|x\|^2/2}$$

Gaussian measure

It is good news if

$$\Phi(e^{i\langle x,\cdot\rangle}) = e^{\text{quadratic in } x}$$

It is the case for Chern-Simons on $\mathbb{R}^3$.

Chern-Simons

3-dim. manifold M say S^3

Lie group G say $SU(2)$

A 1-form on M with values in $L(G)$

$$cs(A) = \mathrm{Tr}\left(A\wedge dA + \frac{1}{3} A\wedge[A\wedge A]\right)$$

This is a 3-form on M.

$$CS(A) = \int_M \mathrm{Tr}\left(A\wedge dA + \frac{1}{3} A\wedge[A\wedge A]\right)$$

$$\int_{\mathcal{A}} f(A)\, e^{ikCS(A)}\, \mathcal{D}A$$

If G is Abelian, the cubic term disappears

$U(1)$ Chern-Simons integrals were worked out by Albeverio and Schäfer using Fresnel integrals.

Non-Abelian CS on $\mathbb{R}^3$

gauge tr. $A = a_0\, dx_0 + a_1\, dx_1 + 0\, dx_2$

$$CS(a_0, a_1) = \langle a_0, -\partial_2 a_1\rangle_{L^2(\mathbb{R}^3)}$$

Fourier transform

$$\int' e^{i\langle b_0,a_0\rangle + i\langle b_1,a_1\rangle} e^{iCS(A)}\, da_0\, da_1 = e^{-i\frac{1}{2}Q^{ax}(b,b)}$$

normalized formal integral

$$Q^{ax}(b,b) = \left\langle (b_0,b_1), \begin{pmatrix} 0 & -\partial_2 \\ \partial_2 & 0 \end{pmatrix}^{-1} (b_0,b_1) \right\rangle$$

Define $\Phi\left(e^{i\langle b_0,\cdot\rangle + i\langle b_1,\cdot\rangle}\right) = e^{-i\frac{1}{2}Q^{ax}(b,b)}$

Φ_{CS} is defined on a space of functions on $\mathcal{A}'$, completion of $\mathcal{A}$.

Can this be related to topological invariants?

Can we evaluate Φ_{CS} on Wilson loop observables?

The answer is most likely, no. It is too much to ask for

Fortunately, a regularization procedure is possible.

Smearing the loop l: 'tube' thickening of l

$$h_{l_\varepsilon}(A)$$

Secondly, deform Q^{ax} by a diffeomorphism ϕ_s of $\mathbb{R}^3$

$$Q^{ax}_{\phi_s}(b) = Q^{ax}(b, (\phi_s)_* b)$$

link L loops $l_1, \dots, l_m$

work out $\Phi_{CS,\phi_s}(L,\varepsilon)$

Atle Hahn:

$$\lim_{\varepsilon\downarrow 0} \lim_{s\downarrow 0} \Phi_{CS,\phi_s}(L,\varepsilon)$$

ϕ_s involves choices related to frames for links

Recent work of Atle Hahn: $S^1 \times \Sigma$.

No axial gauge is possible; instead, use torus gauge fixing (Blau-Thompson).

<u>Why study the CS integral</u>

The CS action provides a toy model for quantum field theory.

Some "real" physical systems have been proposed where the CS action is involved. Ex: gravitation + CS term.

Witten's work on knot invariants.

CS and Chern-Weil

4-dim. mfd W

'Lagrangian density' $\mathrm{Tr}(F^A \wedge F^A)$

Chern-Weil 4-form; its integral over a closed oriented 4-mfd is an integer; a char. class.

$$d\,cs(A) = \mathrm{Tr}(F^A \wedge F^A)$$

$\frac{1}{8\pi^2} CS(A)$ changes by an integer when A is gauge-transformed

$$e^{\frac{1}{8\pi^2} CS(A)} \in U(1)$$

$M^3 = \partial W^4$

Chern-Simons and Yang-Mills

$X^2 = \partial Y^3$

$U(1)$-bundle over the space $\mathcal{A}_X$ of connections over X.

A connection on this $U(1)$-bundle emerges from the CS form.

Pursuing this further leads to a relation between CS action quantizing the system of flat connections on the surface X.

Witten used a formula

$$\mathrm{Tr}(h_\ell(A)) = \int_{\mathcal{G}_\ell} \mathrm{Tr}\left(e^{\int_\ell \mathrm{diag}(A^U)}\right) DU$$