1. Brief history of Maxwell's equations

Magnetism, Its Electricity

1668: Pierre de Maricourt uses magnets
1600: William Gilbert uses magnets
1644: O.M. Grotter, radius, electricity
1697: Newton's theory of gravitation

End of 18th century: founded Franklin, also known as zigzag, produces electricity

1729: Gray conducting, insulating, metal
1745: Duclay takes charge of electricity
1759: Franklin takes the charge of electricity
1805: The French National Convention of Strasbourg

1859: Arnaud, Coulomb, Riou

1860: Michael Faraday, electromagnetic induction, regulation
Historical introduction:

From classical electromagnetism
to gauge theories

1. Brief history of Maxwell's equations

Magnetism Vs Electricity

1269: Piene de Haincort uses magnets
1600: William Gilbert De magnete
1660: Otto von Guericke sulphur Earth
1687: Newton's theory of gravitation

End of XVIIth: academies founded
Francis Hauksbee: 1703 builds machines producing electricity

1730: Grey conducting vs insulating materials
1733: Dufay: two kinds of electricity, vitreous and resinous
1750: Franklin: these two kinds of elec. are of the same nature, conservation of charge
1759: Arpinus (condenser), positive and negative electricity, attraction and repulsion
1740: von Kleist: electric commutation. Musschenbroek: Leyden jar, stone sparks
1771: Cavendish charge stays on the outer surface of conductors.
In a sphere: less than \( \frac{1}{60} \) of the charge stays inside.
Force in \( \frac{1}{r^2} \) at \( r = 2 \) within 2%.

1780: Coulomb studies torsion of threads.
Coulomb's scale.

1785: Mémoires sur l'électricité et le magnétisme.
3 numerical values: \( \frac{1}{r^2} \) law.

1810: Poisson does not try to explain the nature of electricity.

\[-\Delta V = \epsilon = \text{div} \mathbf{E} \]

Electrodynamics

1790: Galvani's experiments in Bologna.
Life frogs.
Sparks produce contraction.
No metals also.
Even one metal, or no metal at all.

Controversy with Volta.

Galvani: animal electricity.
Volta: metallic electricity.

Volta studies torpedo fish.

1800: Volta's pile.

Analogy: pile/magnet, electricity/magnetism.

Ritter: natural philosophy.

1820: Oersted's experiment.

Helical lines of force not a Newtonian theory.

1821: Arago reproduces the experiment in Geneva in front of Ampère.

Biot: force exerted by a magnet on a wire with current.
Thinks that a current is a collection of magnets.

Ampère thinks that a magnet is a bunch of currents.
Idea of interaction of currents.

\[ \text{curl} \mathbf{B} = \frac{\mathbf{j}}{f} \]

1830: After unsuccessful attempts by Colladon in Geneva, and using electro-magnets built by Henry.
Faraday produces current from magnets.

\[ \text{curl} \mathbf{E} = \frac{\partial \mathbf{B}}{\partial t} = 0 \]

The fourth equation:

\[ \text{div} \mathbf{B} = 0 \]

Unclear origin (Kelvin? Another Thomson? Ampère himself?)

1864: Maxwell synthetics.
- Adds the term

\[ \text{curl} \mathbf{B} + \frac{\partial \mathbf{E}}{\partial t} = \frac{\mathbf{j}}{f} \]
for the sake of conservation of charge proposes that light is an electromagnetic phenomenon.

1700 → 1900 uniformity of electricity, magnetism, light, heat

\[
\begin{bmatrix}
\text{div } \mathbf{E} & -\varepsilon_0 \mathbf{E} = \mathbf{J} \\
\text{curl } \mathbf{E} & \frac{\partial \mathbf{B}}{\partial t} \\
\text{div } \mathbf{B} & \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \\
\text{curl } \mathbf{B} & \mu_0 \frac{\partial \mathbf{B}}{\partial t} + \mathbf{k}
\end{bmatrix}
\]

Lorentz force \( q \mathbf{E} + q \mathbf{v} \times \mathbf{B} \)

2. The geometry of Maxwell's equations

Minkowski introduces

\[
\begin{align*}
\mathbf{E} &= E_x \, dx + E_y \, dy + E_z \, dz \\
\mathbf{B} &= B_x \, dy \wedge dz + B_y \, dz \wedge dx + B_z \, dx \wedge dy \\
\mathbf{F} &= \mathbf{B} - \mu_0 \mu_0 \mathbf{E}
\end{align*}
\]

differential forms on space-time \( \mathbb{R}^4 \)

\[ \text{d} \mathbf{F} = 0 \quad \text{homogeneous ME} \]

metric \( dt^2 - dx^2 - dy^2 - dz^2 \), orientation \( dt \wedge dx \wedge dy \wedge dz \)

* Hodge operator \( \ast \mathbf{F}(E, \mathbf{B}) = \mathbf{F}(\mathbf{E}, \mathbf{B}) \)

\[ J = \int d\mathbf{v} \cdot \mathbf{J} + \int t \, d\mathbf{v} \cdot \mathbf{J} + \int v \cdot d\mathbf{v} \cdot \mathbf{J} + \int \mathbf{v} \cdot d\mathbf{v} \cdot \mathbf{J} \]

\[ \text{inhomogeneous ME} \]

\[
\begin{bmatrix}
\text{d} \mathbf{F} = 0 \\
\mathbf{d} \times \mathbf{F} = \mathbf{J}
\end{bmatrix}
\]

Lagrangian form of the equations:

\[ \mathbf{F} = d\mathbf{A} \quad \text{locally} \]

\[ \mathbf{A} \rightarrow \mathbf{A} + d\mathbf{\varphi} \]

\[ L(A) = \frac{1}{2} \mathbf{F} \wedge \mathbf{F} + \mathbf{A} \wedge \mathbf{J} \]

Why is the Lagrangian a function of \( \mathbf{A} \) and not \( \mathbf{F} \)?

- \( \mathbf{A} \) is observable, up to gauge transformation (Belavin-Polyakov)
- \( \mathbf{F} \) does not always determine \( \mathbf{A} \) up to gauge transformations (topology, Gribov ambiguity)

Quantum mechanics of a non-relativistic charged particle

mass \( m \), charge \( q \)

\[ L(\mathbf{r}, \mathbf{v}) = \frac{1}{2} m \mathbf{v}^2 - q \mathbf{V} \cdot \mathbf{A} + q \mathbf{A} \cdot \mathbf{v} \]

\[ \mathbf{A} = - \mathbf{v} dt + \mathbf{A}_x \, dx + \mathbf{A}_y \, dy + \mathbf{A}_z \, dz \]
\[ H(\vec{p}, \vec{p}) = \frac{1}{2m} (\vec{p} - q \vec{A})^2 + qV \]

\[ \vec{p} = m \vec{\pi} + q \vec{A} \]

Schrödinger equation:

\[ (\partial_t + i \vec{H}) \psi = 0 \]

with \[ \vec{H} = -\frac{\hbar^2}{2m} (\nabla^2 - i q \vec{A}) + qV \]

Replace \( A \) by \( A + dq \)

ie \( V \) by \( V - \partial_t \psi \)

\[ \vec{A} \text{ by } \vec{A} + \nabla \psi \]

\( H \) becomes \( H_\psi = -\frac{\hbar^2}{2m} (\nabla^2 - i q \nabla \psi) + q (V - \partial_t \psi) \)

Fact: \( \partial_t + i H_\psi = e^{iq \psi} (\partial_t + i \vec{H}) e^{-iq \psi} \)

A change of gauge modifies the Hamiltonian and must be accompanied by a change of the wave functions.

Principal \( U(1) \)-bundle over space-time (the set of values of norm 1 of the wave functions of a particle of charge 1)

\( A \): connection

\( F \): curvature of \( A \)

Non-abelian gauge theories

\[ P \Sigma G \]

\[ A, F = dA + [A, A] \]

\[ M \]

\[ L(A) = \frac{1}{2} (F \wedge F) + \frac{1}{e} \phi \text{-valued current } 3 \text{-form?} \]

Chromatic hydrodynamics

3. Feynman's path integrals

Particle in a potential

\[ H = -\frac{\hbar^2}{2m} \Delta + V \]

Want to solve \( i \hbar \partial_t \psi = H \psi \), ie

\[ \partial_t \psi = (\alpha \Delta + \beta V) \psi \text{ with } \alpha = \frac{i e^2}{2m}, \beta = \frac{1}{i e} \]

\[ \psi(t, \vec{x}) = (e^{t \Delta + t \beta V}) \psi(0, \vec{x}) \]

Formally,

\[ (e^{a + b}) = \int f(x + i) e^{\frac{ix^2}{2}} dy \]

Also, \( e^{A+B} = \lim_{n \to \infty} (e^{A/n} e^{B/n})^n \).
One finds, formally,

\[ \psi_2(x) = \lim_{n \to \infty} \left( \left( e^{\frac{k}{n^2} \mathbf{r} \cdot \mathbf{A}} \right) \psi_0(x) \right) \]

\[ = \lim_{n \to \infty} \int e^{\frac{k}{n^2} \mathbf{r} \cdot \mathbf{A}} \frac{1}{(2\pi \hbar)^3} \left( \left( e^{\frac{k}{n^2} \mathbf{r} \cdot \mathbf{A}} \right)^* \psi_0(x) \right) \]

\[ = \lim_{n \to \infty} \int e^{\frac{k}{n^2} \mathbf{r} \cdot \mathbf{A}} + \frac{k}{n^2} \mathbf{r} \cdot \mathbf{A} + \cdots + \frac{k}{n^2} \mathbf{r} \cdot \mathbf{A} \cdot \psi_0(x) \]

\[ = \int \frac{1}{(2\pi \hbar)^3} e^{\frac{k}{n^2} \mathbf{r} \cdot \mathbf{A}} \frac{1}{\sqrt{4\pi \hbar}} \psi_0(x) \]

\[ \mathcal{L} = \int e^{\frac{i}{\hbar} \int_0^t \left( -\frac{1}{2} \dot{q}^2 + V(q(t)) \right) dt} \psi_0(q(t)) \mathcal{D}q \]

\[ = \int e^{\frac{i}{\hbar} \int_0^t \left( -\frac{1}{2} \dot{q}^2 - V(q(t)) \right) dt} \psi_0(q(t)) \mathcal{D}q \]

The amplitude of presence at point z is

\[ \int e^{\frac{i}{\hbar} \int_0^t \frac{\mathbf{g}(t)^2}{2} + V(q(t)) dt} \psi_0(q(t)) \mathcal{D}q \]

\[ \{ q \in \mathbb{R}, q(0) = x, q(t) = z \} \]

In the context of gauge theories, these integrals become

\[ \int e^{\frac{i}{\hbar} S(A)} \mathcal{D}A \]

Some amount of A

Stochastic quantisation: make sense of

\[ \mathcal{D}A = \frac{1}{2} e^{-\frac{1}{2} S(A)} \mathcal{D}A \]

as a probability measure.
Gauge fields mediate interactions between fundamental constituents.

EM field is described through potential and vector potential, $A$ and $A'$, respectively.

Field strength vector in force:

$F_{EA}$

$F = B \times E$

$B = \nabla \times A$, $E = -\nabla A'$

The force exerted by the field on a point charge $q$, usually $\mathbf{a}$, adjacent objects is $\mathbf{F} = q \mathbf{E}$, and in Mechanics - orthogonal to $\mathbf{v}$.

Force: mass $m$ x rate of change of momentum (which is a vector).

Lorentz force law:

$\mathbf{F} = q \mathbf{E} + q \mathbf{v} \times \mathbf{B}$
Two-dimensional Yang-Mills

Review of EM: Maxwell equations, potentials, force law, action

Gauge fields mediate interaction between fundamental constituents

EM field is described through a potential, a 1-form $A$ on 4-dim. spacetime $\Sigma$.

Field strength: a 2-form

$$F = \text{d}A$$

$$F = B - \text{d}t \wedge E$$

$$B = B_z (dz \wedge dx, dz \wedge dx, dz \wedge dy)$$

$$E = E_z (dz, dy, dx)$$

The force exerted by the field on a point charge $e$ moving at velocity $v$ depends linearly on $\xi$ and is Minkowski-orthogonal to $v$.

Force: mass $\times$ rate of change of momentum (which is a vector).

Lorentz force law:

$$\text{force} = e \times v \times F$$
Maxwell equations
\[
\begin{align*}
\mathbf{d}F &= 0 \\
\mathbf{d} \times \mathbf{F} &= \mu_0 \mathbf{J}
\end{align*}
\]

The electromagnetic field has the ability to do work on charges and currents. It thus has energy. Maxwell worked out the energy density of the field

\[ u = \frac{\varepsilon_0}{2} (1 + c^2 \| \mathbf{B} \|^2) \]

Momentum of the field: Poynting vector

\[ \mathbf{P} = \frac{\mu_0}{2} \mathbf{E} \times \mathbf{B} \]

Write \( \mathbf{A} = -\frac{1}{c} \phi \, dt + \mathbf{A} \cdot (dx, dy, dz) \)

\[ L_{\text{EM}} (\phi, \mathbf{A}) = \int_{\mathbb{R}^3} \left( \frac{\varepsilon_0}{2} \left[ (1 - c^2 \| \mathbf{B} \|^2) - (\mathbf{E} - \mathbf{A} \times \mathbf{A}) \right] \right) \, d\mathbf{r} \]

\[ \mathbf{E} = -\nabla \phi - \mathbf{A} \], \[ \mathbf{B} = \nabla \times \mathbf{A} \]

\[ S_{\text{EM}} = \int_{\mathbb{R}} L_{\text{EM}} (\phi, \mathbf{A}) \, dt \]

This works out as

\[ S_{\text{EM}} (\mathbf{A}) = \int_{\Sigma} L_{\text{EM}} (\mathbf{A}) \]

\[ L_{\text{EM}} (\mathbf{A}) = -\frac{c^2 \varepsilon_0}{2} \mathbf{F} \times \mathbf{F} - \mathbf{A} \times \mathbf{J} \]

(The sign matters if one is going to look for the Hamiltonian and understand it as energy)

(There is a formalism for non-orientable spacetimes, using densities)

Quantization for a particle in the EM field and gauge theory.

Quantum mechanics puts a restriction on the behavior of the field strength \( E \)

\[ \frac{e}{\hbar} \int F \cdot \, dt \in \mathbb{Z} \]

for every closed oriented 2-manifold sitting in spacetime. (Dirac monopole charge quantization condition).

This is precisely the condition for \( i \frac{e}{\hbar} F \) to be the curvature for a connection.
This geometric feature of the electromagnetic field makes it a gauge theory in a theory of connections.

Gauge invariance.

\[ \psi \text{ Schrödinger wave function for a particle change } e \text{ in potential } A \]

Compute with

\[ \psi \quad \text{ and } \quad A \]

or

\[ \psi_0 = e^{i \theta} \psi \quad \text{ and } \quad A^0 = A + d\theta \]

θ: 'angle of rotation' in 'charge space'.

Then \( (\hat{E} - i \frac{e}{c} A_j) \psi \mapsto (\hat{E} - i \frac{e}{c} A_j^0) \psi_0 \)

\[ F = dA = dA^0 \]

Since it is \( i \frac{e}{c} A \) that is actually a connection form (has dimension \( [L^{-1}] \)), let us rewrite this as \( A \) and keep

\[ S_{\text{em}} (A) = -k \sum F^A \wedge \ast F^A \]

Nucleons

1932: Heisenberg suggested that the proton and the neutron could be viewed as different states of the same particle.

"Isospin": they are different states of isospin of the same entity.

As far as isospin is concerned, the nucleon wave function \( \psi \) is something on which \( SU(2) \) matrices should be able to act. (Just as \( U(1) \) acts on the traditional wave function for an electrically charged particle).

Yang-Mills: extending to isotopic gauge invariance

\[ \psi \text{ the Schrödinger wave function for a particle with isotopic spin moving in the field that propagates the interaction between particles with isotopic spin.} \]

Yang and Mills (1954) suggested a local gauge invariance principle

\[ \psi \quad \text{ and } \quad A \]

or

\[ \psi_0 = U \psi \quad \text{ and } \quad A^0 = U A U^{-1} (dU) U^{-1} \]

U: \( SU(2) \)-valued function on spacetime.
A: 1-form with values in 2x2 skew-Hermitian matrices.

Field strength
\[ F^A = dA + A \wedge A \]

Classical field configurations are extrema of the Yang-Mills action
\[ S_{YM}(A) = \frac{1}{2g^2} \int_{\Sigma} \text{Tr}(F_A \wedge *F_A) \]

Euler-Lagrange equations: YM equations.

Gauge fields as connection forms

1-form \( A \) on \( \Sigma \) with values in the Lie algebra \( \mathfrak{g}(G) \) of a compact Lie group \( G = U(N) \).

\( v \in T_{v}X \Rightarrow A(v) \in \mathfrak{g}(G) \)

Connection, or a gauge field

Set of all connections is an infinite-dim. vector space

\[ \mathcal{A} \]

Metric structure \( \text{Ad-inv.} \)

\[ \langle A, B \rangle = \int_{\Sigma} \langle A, B \rangle_{\mathfrak{g}(G)} \text{dvol} \]

Wilson loop variables

A gauge transformation \( \phi : \Sigma \rightarrow G \)

\( G \) group under pointwise mul.

\[ A^\phi = \phi^{-1} A \phi + \phi^{-1} d\phi \]

\( \mathcal{G} / G \) quotient space

\( \mathcal{G}_0 \) : those \( \phi \in \mathcal{G} \) s.t. \( \phi(0) = I \) for some basepoint 0

Parallel transport

Connection \( A \), smooth path \( c : [0, 1] \rightarrow \Sigma \)

\[ [0, 1] \rightarrow G \]

\[ t \mapsto g_t \]

\[ \frac{dg_t}{dt} = -A(c'(t)) g_t \]

\( g_0 = I \)

\( g_t \in G \) is called the holonomy of \( A \) around \( c \)

\[ \text{Ad}_{c}(A) = h(c; A) = g_t \]

\[ \text{Tr}(h_c(A)) \] Wilson loop variable
Non-abelian gauge theory: quantum functional integral

Quantizing the gauge field itself requires (in one approach) using a functional integral measure

$$\frac{1}{\frac{1}{2g}} e^{-\text{Sym}(A)} DA$$

$$\frac{1}{\frac{1}{2g}} \int A f(A) e^{-\text{Sym}(A)} DA$$

Simplifying the quartic

$$\text{Sym}(A) = \| dA + A \otimes A \|^2$$

which is quartic in $A$ and the problem is difficult.

Work with $A_{/g}$: leads to a simplification.

YM on $\mathbb{R}^2$ is Gaussian

$$A_{/g_0} = A = A_x dx + A_y dy$$

with $A_y = 0$.

$$F^A = dA + 0$$

This makes our functional integral measure have a very convenient appearance:

$$\frac{1}{\frac{1}{2g}} e^{-\frac{1}{2g} \| F \|^2} DF$$

No useful form of Lebesgue in infinite dimensions.

But useful and very useful Gaussian.

The Yang–Mills measure for gauge theory on $\mathbb{R}^2$ is Gaussian measure on functions

$$f: \mathbb{R}^2 \to L(G)$$

Stochastic geometry.

Consider a path

$$c: [0,1] \to \mathbb{R}^2$$

$$t \mapsto (t, y(t))$$

$$dg_t = -A(c'(t)) g_t dt$$

Now that $A$ is stochastic, this can be reinterpreted as a Stratonovich SDE.

If $c$ is a nice loop in $\mathbb{R}^2$, the holonomy

$$h_c(A)$$

as a function of the stochastic $A$, is a random variable with values in $G$.

Its distribution has density

$$Q_{g^2/16}(x)$$

where $1/16$ is the area enclosed by the loop $c$.

$$\Delta Q_T(x) = \frac{1}{2} \Delta Q_T(x).$$
Loop expectation values.

Theorem 1. c simple closed loop in $\mathbb{R}^2$ enclosing area $S$: holonomy $h_c$ distributed according to $Q_{\mathcal{G}^1}(x) \, dx$.

Non-overlapping loops are mutually independent.

Single loop for $U(N)$

$$\mathbb{E}[T_{r_0}h_c] = e^{-N\frac{\pi}{4}S}$$

Compact manifolds

2-dim. Riem. mfd $\Sigma$

Build $\Sigma$ from a disk by identifications

Conditional probability measure: holonomy along this arc equals holonomy along that arc.

Distribution of the random variables $h_{c_1}, \ldots, h_{c_n}$

determine all that is of interest for the YM measure.

Stochastic holonomy fields

$\Sigma$ 2-dim. Riem. mfd

$L_0(\Sigma)$ set of rectifiable loops $h_c$ holonomy, with f.d. distr.

Loop expectation value on the sphere $c$ on the sphere $S^2$

$h_c$ Brownian bridge

$$\mathbb{E}[f(h_c)] = \frac{1}{Q_{\mathcal{G}^1}(c)} \int f(x) Q_{\mathcal{G}^1}(x) Q_{\mathcal{G}^1}(x^*) \frac{dx}{G^2}$$

General configuration of loops

Statistical physics flavor.

Graphs on the surface.

Freezing the measure

$$d\mu_g(A) = \frac{1}{2g} e^{-\frac{1}{2g} \int F_{\alpha\beta}^2} \, DA$$

$g^2 \to 0$ $\mu_g$ freezes on the space of flat connections

Large $N$ limit

$U(N), N \to \infty$, holding $N^2$ constant.
A connection with random matrix theory.

$U(N)$ Yang-Mills theory on $\mathbb{R}^2$

$\mathbb{R}^2$ $U(N)$ Yang-Mills measure

$$d\mu_{YM}(A) = \frac{1}{2\pi^2} e^{-\frac{1}{2g^2} S_{YM}(A)} \, DA$$

1 loop $h_l(A) \in U(N)$ holonomy along $\gamma$

A random scalar under $\mu_{YM}$:

$$h_l(A) = h_{l_1} \in U(N)$$ $U(N)$-valued r.v.

1. $h_{l_1} \sim Q_{g^2} \mu_{YM}$

2. Heat kernel, Haar meas.

$= \text{dist. of } B_{g^2} \in \text{BM on } U(N)$

3. $h_{l_1}, h_{l_2}, h_{l_3}$ independent

4. $h_l h_{l_1} h_{l_2} = h_{l_1} h_{l_2} h_l$

"Brownian motion on $U(N)$ indexed by loops, with area playing the role of time, and with a property of multiplicativity"
The "free group" of loops
loops can be concatenated

notion of cancellation (backtracing)
the group of loops "is" free on the set of elementary loops (lassos)
any reasonable loop can be written as a word in non-overlapping lassos

Ex:

more complicated, but algorithmically and analytically tractable

Wilson loop expectations

What are the numerical quantities of interest?

Gauge-invariant quantities

\[(A, l) \rightarrow h(l, A)\]

\[(A^0, l) \rightarrow h(l, A^0) = U(l(0)) h(l, A) U(l(0))\]

Basic GI quantity: \(h_{\mathbf{c}_1}, \ldots, h_{\mathbf{c}_n}\) up to simultaneous conjugation.

Unitary matrix \(U\) up to conjugation:

spectrum of \(U\)

\([U_{\mathbf{c}_1}, \ldots, U_{\mathbf{c}_n}]\) up to simultaneous conjugation:

spectra of \([U_{\mathbf{c}_1}, \ldots, U_{\mathbf{c}_n}]\) + relative pos of eigenspaces

Theorem: If for every word \(w\) in \(n\) letters and their inverses one has

\[\text{Tr}(w(U_{\mathbf{c}_1}, \ldots, U_{\mathbf{c}_n})) = \text{Tr}(w(U^t_{\mathbf{c}_1}, \ldots, U^t_{\mathbf{c}_n}))\]

then there exists \(V \in \text{U}(N)\) s.t.

\[V_{\mathbf{c}_i} = U_{\mathbf{c}_i} V^{-1}\]

The basic numerical quantity is

\[\text{Tr}(w(h_{\mathbf{c}_1}, \ldots, h_{\mathbf{c}_n})) = \text{Tr}(h_{\mathbf{c}_1} w(h_{\mathbf{c}_1}, \ldots, h_{\mathbf{c}_n}))\]

Wilson loop expectations:

\[E[\text{Tr}(h_{\mathbf{c}_1}) \ldots \text{Tr}(h_{\mathbf{c}_n})]\]
Using the "free group" structure of loops
on $\mathbb{R}^2$ (with a lot of independence),
this becomes

$$E \left[ \text{Tr}(w_n(B_e^{(n)}, \ldots, B_e^{(p)})) \ldots \text{Tr}(w_n(B_e^{(n)}, \ldots, B_e^{(p)})) \right]$$

words in letters and their inverses

Brownian motion on $U(N)$

Laplacian on $U(N)$

$\Delta_X = \text{NTr}(X^*Y) = -N \text{Tr}(XY)$

ONB:

$$\begin{pmatrix} \frac{1}{\sqrt{2N}} \epsilon_{1} \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{N}} \epsilon_{N} \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{N}} \epsilon_{N} \end{pmatrix}$$

$$\begin{pmatrix} X_k \end{pmatrix}_{k=1}^{N^2}$$

$g = \frac{1}{N}$

$\Delta f(u) = \sum_{k=1}^{N^2} \frac{d^2}{dt^2} |_{t=0} f(ue^{\frac{t}{N}})$

$E[f(u)] = \text{Tr}(u)$

$E[\Delta f(u)] = \sum_{k=1}^{N^2} \text{Tr}(UX_k^2) = \text{Tr}(U \sum_{k=1}^{N^2} X_k^2)$

$\sum_{k=1}^{N^2} X_k^2 = -IN$

$\Delta Tr = -Tr$

Brownian motion on $U(N)$: Markov process on $U(N)$ with generator $\frac{1}{2} \Delta$, issued from $I_N$. $$(B_t)_{t \geq 0}, B_0 = I_N.$$ $f: U(N) \to \mathbb{R}$

$$\frac{\partial}{\partial t} E[f(B_t)] = E[\frac{1}{2} (\Delta f)(B_t)]$$

$E[f(B_0)] = f(I_N)$

Example $t = \frac{1}{2} Tr$

$E[Tr(B_t^2)] = e^{-\frac{t}{2}}$

Brownian motion on $U(N)$ and walk on $S_n$.

$E[\Delta f(B_t^2)] = ?$

$\frac{\partial}{\partial t} f(u) = \text{Tr}(u^2)$

$\frac{\partial}{\partial t} f(u) = \sum_{k=1}^{n^2} \frac{d^2}{dt^2} |_{t=0} \text{Tr}(uX_k^2)$

$\Delta f(u) = \sum_{k=1}^{n^2} \text{Tr}(uX_k^2) = \text{Tr}(u \sum_{k=1}^{n^2} X_k^2)$

$= 2 \text{Tr}(u \sum_{k=1}^{n^2} X_k^2) + 2 \sum_{k=1}^{n^2} \text{Tr}(uX_k^2)$
$\sigma$ has 1 more cycle if $ij$ are in the same cycle of $\sigma$

\[ \frac{1}{2} \Delta_{v_{\sigma}} (u) = -\frac{n}{2} \prod_{v_{\sigma}} (u) - \sum_{ij, \text{ same cycle}} \prod_{v_{ij}} (u) \]

\[ -\frac{1}{N^2} \sum_{ij, \text{ different cycles}} \prod_{v_{ij}} (u) \]

The $n!$ functions

$$(t \mapsto \mathbb{E} [\prod_{v_{\sigma}} (B_t^{\sigma})])_{\sigma \in S_n}$$

satisfy a first-order linear diff. eq.

This system can be solved for finite $N$.

As $N \to \infty$, the system becomes simpler.

Theorem: \( \mathbb{E} [t_i (B_t^{\omega_i}) \ldots t_i (B_t^{\omega_i})] = \mathbb{E} [t_i (B_t^{\omega_i})] \ldots \mathbb{E} [t_i (B_t^{\omega_i})] + o\left( \frac{1}{N^2} \right) \)

(factorization property)

\[ \lim_{N \to \infty} \mathbb{E} [t_i (B_t^{\omega_i})] = e^{-\frac{t}{T^2}} \sum_{k=0}^{\infty} \left( \frac{t}{T^2} \right)^k \frac{(n)}{k!} n^{k-1} \]

(Wigner's limit $Z_1, \ldots, Z_N$ iid $\mathcal{N}(0,1)$)

$M = \sum_{k=1}^{N} Z_k (iX_k)$ Gaussian Hamiltonian matrix

The eigenvalues of $M$ are dist. according to

\[ \sigma(x) = \frac{1}{2\pi t} \sqrt{4t - x^2} \frac{1}{\sqrt{[2t, 2t]}} (x) \]

Here there is a prob. meas. on $U$,

where $U = \{ z \in \mathbb{C} : |z| = 1 \}$, s.t.

the eigenvalues of $B_t$ are (almost)

distributed according to $\beta_t$.

\[ \text{Suff}(\beta_t) = \begin{cases} 
\emptyset & \text{if } t < 4 \\
U & \text{if } t \geq 4
\end{cases} \]

Small $t$: $\beta(t) \approx 2t^4$.
Three-dimensional Chern-Simons theory

Study of infinite dimensional integrals of the type

\[ \int_A f(A) \ e^{-\beta S(A)} \ DA \]

is rich with challenges and questions arising both from physics and mathematics. Cases where this integral represents a quantity of geometric or topological meaning that can be understood in other ways.

Sometimes a limit \( \beta \to 0, \infty \) is involved.

One usually treats the left side formally and extracts insights from this formal integral.

- Perturbative
- Non-perturbative

Yang-Mills

\[ \frac{1}{2g} \int_A f(A) \ e^{-\frac{1}{2g^2} \ Sym(A)} \ DA \]

Chern-Simons

\[ \int_A f(A) \ e^{ikCS(A)} \ DA \]
At a heuristic level, $\Omega A$ requires a choice of metric (or something similar) on $\mathcal{A}$.

V infinite dim. space
\[ \int \phi(x) e^{-\beta S(x)} \, dx \]

is a linear functional
\[ \phi : \phi \mapsto \phi(f) \]

Formal calculations specify what $\phi(f)$ should be for some good class of functions.

$\phi$ should have continuity properties
$\phi$ might come from integration w.r.t. a measure, or be a distribution

V infinite dim. real vector space
\[ \phi(e^{i\langle x, \cdot \rangle}) = e^{-\frac{1}{2}||x||^2} \]

Gaussian measure

It is good news if
\[ \phi(e^{i\langle x, \cdot \rangle}) = e \text{ quadratic in } x \]

It is the case for Chern-Simons on $\mathbb{R}^3$.

Chern-Simons

3-dim. manifold $M$ say $S^3$
Lie group $G$ say $SU(2)$
$A$ 1-form on $M$ with values in $L(G)$
\[ \omega(A) = Tr(A \wedge dA + \frac{1}{3} A \wedge [A \wedge A]) \]

This is a 3-form on $M$.

\[ \text{CS}(A) = \int_M Tr(A \wedge dA + \frac{1}{3} A \wedge [A \wedge A]) \]

If $G$ is Abelian, the cubic term disappears

$U(1)$ Chern-Simons integrals were worked out by Albeverio and Schäfer using Fresnel integrals.

Non-Abelian CS on $\mathbb{R}^3$

gauge tr. $A = a_0 \, dx_0 + a_1 \, dx_1 + 0 \, dx_2$

\[ \text{CS}(a_0, a_1) = \langle a_0, -\frac{2}{3} a_1 \rangle_{L^2(\mathbb{R}^3)} \]

Fourier transform
\[
\int e^{i\langle b_0, a_0 \rangle + i\langle b_1, a_1 \rangle} e^{iCS(A)} da_0 da_1 = \\
e^{-i \frac{1}{2} Q^{ax}(b, b)}
\]

normalized formal integral

\[Q^{ax}(b, b) = \langle (b_0, b_1), (0, -\partial_z)^{-1}(b_0, b_1) \rangle\]

Define \(\phi(e^{i\langle b_0, \cdot \rangle + i\langle b_1, \cdot \rangle}) = e^{-i \frac{1}{2} Q^{ax}(b, b)}\)

\(\phi_{CS}\) is defined on a space of functions on \(A\), completion of \(A\).

Can this be related to topological invariants?

Can we evaluate \(\phi_{CS}\) on Wilson loop observables?

The answer is most likely, no. It is too much to ask for

Fortunately, a regularization procedure is possible.

Shrinking the loop \(l\): 'tube' thickening of \(l\)

\(\phi_{(A)}\)

Secondly, deform \(Q^{ax}\) by a diffeomorphism

\(\phi_{3}\) of \(\mathbb{R}^3\)

\[Q_{\phi_{3}}(b) = Q^{ax}(b, (\phi_{3})_*(b))\]

Link L loops \(l_1, \ldots, l_m\)

Note out \(\phi_{CS}, \phi_{3}(L, \Sigma)\)

Atle Hahn:

\[\lim_{\delta \to 0} \lim_{\delta \to 0} \phi_{CS}, \phi_{3}(L, \Sigma)\]

\(\phi_{3}\) involves choices related to frames for links

Recent work of Atle Hahn: \(S^1 \times \Sigma\).

No axial gauge is possible; instead, use torus gauge fixing (Blau-Thompson).

\[\text{Why study the CS integral}\]

The CS action provides a toy model for quantum field theory.

Some "real" physical systems have been proposed where the CS action is involved. Ex: gravitation + CS term.

Witten's note on knot invariants.
CS and Chern-Weil

4-dim. mfd \( W \)

'\text{Lagrangian density}' \( \text{Tr}(F^\Lambda \wedge F^\Lambda) \)

Chern-Weil 4-form; its integral over a closed oriented 4-mfd is an integer; a char. class.

\[ d \text{cs}(A) = \text{Tr}(F^\Lambda \wedge F^\Lambda) \]

\[ \frac{1}{8\pi^2} \text{CS}(A) \text{ changes by an integer} \]

when \( A \) is gauge-transformed

\[ e^{\frac{i}{8\pi^2} \text{CS}(A)} \in U(1) \]

\[ M^3 = \mathcal{W}^4 \]

Chern-Simons and Yang-Mills

\[ X^2 = \mathcal{Y}^3 \]

\( U(1) \)-bundle over the space \( A_x \)

of connections over \( X \).

A connection on this \( U(1) \)-bundle emerges from the CS form.

Pursuing this further leads to a relation between CS action quantizing the system of flat connections on the surface \( X \).